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# Modeling of Composite Beams and Plates for Static and Dynamic Analysis

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## Summary of Work Done Under Grant Sponsorship

The main purpose of this research has been to develop a rigorous theory and corresponding computational algorithms for through-the-thickness analysis of composite plates. This type of analysis is needed in order to find the elastic stiffness constants for a plate and to post-process the resulting plate solution in order to find approximate three-dimensional displacement, strain, and stress distributions throughout the plate. This also requires the development of finite deformation plate equations which are compatible with the through-the-thickness analyses.

After about one year's work, we settled on the variational-asymptotical method<sup>1</sup> (VAM) as a suitable framework in which to solve these types of problems. VAM was applied to laminated plates with constant thickness in the work of Atilgan and Hodges<sup>2</sup>. The corresponding geometrically nonlinear global deformation analysis of plates was developed by Hodges, Atilgan, and Danielson<sup>3</sup>. A different application of VAM, along with numerical results, was obtained by Hodges, Lee, and Atilgan<sup>4</sup>. An expanded version of this last paper<sup>5</sup> has been submitted for publication in the *AIAA Journal*. (Copies of these papers have been delivered to Mr. Hinnant.) One last paper was just completed<sup>6</sup> and a copy is included as an appendix to this report. Summaries of the progress in the various categories we worked on follow. Technical details are in the papers.

### Work on Finite Deformation of Plates

In Ref. 3, a set of kinematical and intrinsic equilibrium equations are derived for large deflection and rotation but with small strain. This work showed that the drilling type rotation is not an independent variable in plate theory. If it is to be included in plate equations at all, there must be a constraint enforced between it and its definition in terms of other plate variables. Unless this constraint is enforced, the theory including the rotation as a separate variable is not valid. The main contribution of this paper is a complete set of geometrically exact strain-displacement relations for large deformation of plates.

Also, the relationship between the drilling rotation and the other kinematical variables gives new insight into the drilling moment and its role in beam-plate connectivity. An applied drilling moment at a point on a plate is not resisted at all by the plate. Such a moment, in order to have any physical resistance from the plate, must be applied over a finite area. Other than this, a point drilling moment can only be resisted by a plate if the plate model is derived from couple-stress elasticity<sup>7</sup>. Further study on this problem was shelved, because of the need to interpret the drilling rotational degrees of freedom in certain plate and shell finite element formulations. There is an apparent conflict between the theoretical result of a plate having zero stiffness in response to a point drilling moment on one hand, and the finite element results found in the literature for the response of plates to point drilling moments. The papers in which these results appear are of several researchers, some of whom are highly regarded by the international community and by the principal investigator. A dialog has been initiated on this subject with some of these people, but to date we have no resolution to this question.

## Work on Modeling Laminated Plates

By modeling here we mean the calculation of elastic constants for a two-dimensional model based on known material constants for the three-dimensional body. This can only be done approximately, of course, and asymptotical methods are natural. In Ref. 2, the "first approximation" is exactly the same as classical laminated plate theory. The "second approximation" takes transverse shear deformation into account and is asymptotically correct only for plates with certain restrictions in their construction. To remove the restrictions one must "kill" certain interaction terms in the strain energy, but the means for doing so for general laminated plates were not given in that paper. Indeed, the means for doing this were unknown to us at that time.

The development in Refs. 4 and 5 includes transverse shear in the "first approximation" and is stopped there. Results from this theory were compared, for the cylindrical bending case, with results from the exact solution of Pagano for both cross-ply<sup>8</sup> and shear-coupled<sup>9</sup> laminated plates. The resulting theory, termed a "neo-classical" theory (NCPT), is at least as good as classical plate theory (CPT) in every case; for most cases NCPT is superior to CPT. For example, NCPT does a much better job on calculation of plate (two-dimensional) displacement than CPT. Also, many three-dimensional quantities are calculated much more accurately than could be achieved with CPT. Results for several different types of plates may be found in Ref. 5.

As pointed out in Ref. 5, there is a need to develop higher approximations in order to (1) improve the overall performance of the theory for applications to thick plates and (2) improve the accuracy of predicted transverse strains and stresses, especially in situations when integration of the three-dimensional equilibrium equations cannot be accomplished, such as in the geometrically nonlinear case.

## Work on a Higher Approximation for NCPT

In order to improve the asymptotical accuracy of NCPT one needs to introduce "degrees of freedom" of the normal line element which will eliminate or "kill" interaction terms of the type identified in Ref. 2. In our work during the last reporting period we applied Sutyrin's eigen-principle (described in the last report) to develop a refined theory for laminated plates of constant thickness. This theory includes CPT as a special case and even in its most elementary extension should surpass NCPT in predictive capability. Furthermore, because it is based on a variable number of "degrees of freedom" for the normal line element, it should allow users of future finite elements based on this theory to decide which of these degrees of freedom they want in their models. In principle, one can approach the accuracy of three-dimensional elasticity to any degree desired. The theory is completed as far as the derivation of the stiffness model based on eigenfunctions associated with a certain Sturm-Liouville operator. A computer code has been written and validated for the calculation of these eigenfunctions. Another code is presently under development which will use these eigenfunctions to build the stiffness model. For the details of the stiffness model, see Ref. 6.

Depending on what funding we can find, in the future we intend to generate results from this new theory for the cylindrical bending problem. This will allow us to inexpensively evaluate the number and types of normal line element degrees of freedom required for accurate solutions. The last step would then be to build a prototype plate finite element. We believe that this element would be far superior to any extant plate finite elements. We would want to work with someone who already has experience in two-dimensional finite elements. The strength of our element would be in the theoretical foundation, not in the finite element technology itself. Dr. Sutyrin also has devised a means by which a discretized model based on three-dimensional finite elements can be reduced directly to plate elements. While this should be equivalent to our present semi-analytical approach, it may prove to be more convenient in certain contexts.

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# A Variable-Order Laminated Plate Theory Based on the Variational-Asymptotical Method

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## Abstract

The variational-asymptotical method is a mathematical technique by which the three-dimensional analysis of laminated plate deformation can be split into a linear, one-dimensional, through-the-thickness analysis and a nonlinear, two-dimensional, plate analysis. The elastic constants used in the plate analysis are obtained from the through-the-thickness analysis, along with approximate, closed-form three-dimensional distributions of displacement, strain, and stress. In this paper, a theory based on this technique is developed which is capable of approximating three-dimensional elasticity to any accuracy desired. This theory is not developed using any of the usual approaches of laminated plate theory. That is, it is not based on any power series expansion through the thickness, nor is it based on introduction of a set of variables which describe displacement in separate layers of laminated plates. Rather, the asymptotical method allows for the approximation of the through-the-thickness behavior in terms of the eigenfunctions of a certain Sturm-Liouville problem associated with the thickness coordinate. These eigenfunctions contain all the necessary information about the nonhomogeneities along the thickness coordinate of the plate and thus possess the appropriate discontinuities in the derivatives of displacement. The theory is presented in this paper along with numerical results for the eigenfunctions of various laminated plates.

## Introduction

For aerospace structures, laminated composite materials provide excellent opportunities for structural simplicity as well as elastic couplings for potential optimization of design criteria. Although plates made of such materials have been used for some time in a variety of engineering applications, simple and efficient methods for analyzing plates with anisotropy and nonhomogeneity are still needed. This is partly due to rapid changes taking place in manufacturing technology for composite materials and partly to ever-increasing demands for accuracy and efficiency. Much of what is done is based on classical plate theory (CPT) which,

although adequate for many engineering applications, has well known limitations due to the Kirchhoff hypothesis.

## Background

Many attempts have been made to improve classical theory by taking into account non-classical behavior such as transverse shear deformation and transverse normal stresses. From the time laminated fiber-reinforced composites were first introduced, numerous works have been published, the objectives of which include the improvement of CPT for laminated plate applications. This subject is discussed at length in review papers.<sup>1,2</sup> There are two main classes of methods for improving plate theory found in the literature: (1) *Power Series Methods*: expansion of the displacement field variables into higher-order power series in the thickness coordinate; and (2) *Layerwise Variables Methods*: incorporation of separate sets of displacement variables for each layer. Both of these methods have known shortcomings. For example, no power series expansion can possibly render accurate results for quantities which may possess discontinuities, such as certain components of strain and stress in laminated plates. The layerwise variables methods rely on a significant increase in the number of unknowns, a number which depends on the number of layers in the plate. A third method has received some attention in recent years, which involves an assumed displacement field with discontinuities allowed in through-the-thickness derivatives.<sup>3,4,5</sup> There is no question that this method yields excellent results in some cases, but it lacks a systematic basis for choosing the displacement functions, and it does not yield an asymptotically correct result in general.

Ref. 6 undertook a quite different approach. It does not involve a power-series expansion through the plate thickness, nor does it involve layerwise unknowns. Rather, the three-dimensional energy of a laminated plate was approximated following Berdichevsky's variational-asymptotical methodology.<sup>7</sup> Normally asymptotical methods are employed for analytical developments, but here such a method was used in a sort of semi-analytical approach. Namely, the theory leads to a Reissner-like plate theory, along with a set of elastic constants; it also provides a set of influence functions from which approximate three-dimensional displacement, strain, and stress fields can be determined once the plate equations are solved. The plate equations can be solved by any method desired, such as a two-dimensional finite element method. The analysis was restricted to laminated plates for which each lamina exhibits monoclinic material symmetry about its middle surface. Their first approximation is asymptotically correct for this case and coincides with classical laminated plate theory. However, their

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second approximation is asymptotically correct only when each element of the reduced stiffness matrix  $\bar{Q}$  (see Jones<sup>8</sup>) is constant through the thickness of the entire plate. Although the theory is not asymptotically correct otherwise, it was intended for application to laminated composite plates. The limitation of their work stems from a term in the strain energy which was neglected. A suitable method to make this term vanish rigorously is needed in order to make the theory asymptotically correct, but was not given.

A similar, but somewhat improved, approach was undertaken by Refs. 9 and 10, in which the estimation procedure of Ref. 6 was slightly modified to include transverse shear terms in the first approximation. Plates with cross-ply stacking sequences under cylindrical bending were taken as example problems in Ref. 9. In a later extension of this work,<sup>10</sup> plates with arbitrary stacking sequences undergoing cylindrical bending were taken as example problems. The material configurations of these latter plates are not as simple as those of bi-directional plates, because of the influence of the coupling of transverse shear terms. The distributions of three-dimensional displacement, strain, and stress were investigated for both cases by comparing the corresponding three-dimensional exact elasticity solution.<sup>11,12</sup> The theory of Refs. 9 and 10, termed the "neo-classical" plate theory (NCPT), was shown to be more accurate than CPT for thick, laminated plates; also, it was shown to yield results which are somewhat better than those of the theory of Ref. 6.

Still, there were results reported for which the correlation of NCPT with the exact solution is not good. For example, when shear coupling exists, NCPT shows significantly better correlation with the exact solution than for the bi-directional cases. It is necessary, then, to extend the validity of the theory to a higher approximation. Although it is not discussed in Refs. 9 and 10, such an extension requires that certain interaction terms vanish. These terms are analogous to the one neglected in Ref. 6. For this reason a method for generalization of the theory of Refs. 9 and 10 has been developed, and that is the subject of the present paper.

## Present Approach

The essence of the new approach, which guarantees disappearance of the interaction terms discussed above, is the introduction of new "degrees of freedom" into the three-dimensional displacement field. We are not using the term "degrees of freedom" in its usual sense. Here we mean plate (i.e., two-dimensional) displacement variables which are associated with a particular deformation mode of the normal line element through the thickness. The shape functions of these new degrees of freedom are chosen as the eigenfunctions associated with a certain Sturm-Liouville problem formulated by Sutyurin.<sup>13</sup> By choosing the associated warping displacement to be orthogonal to the shape functions for each of the new degrees of freedom, the displacement field is uniquely defined. The choice of shape functions from these

eigenfunctions guarantees that any additional warping induced by the new degrees of freedom is of a higher order relative to the new degrees of freedom themselves. The order of these new degrees of freedom relative to the strain depends on the loading and the material constants. After obtaining the eigenvectors by using a one-dimensional finite element method, the plate elastic constants can be obtained through the variational-asymptotical method. Utilizing global deformation equations along with this resulting plate constitutive law, we will complete the formulation of the theory.

In this paper we first provide the details for the kinematics of plate deformation in terms of classical plate displacement variables and the new degrees of freedom. The strain field needed to develop a geometrically nonlinear plate theory is written in terms of these displacement variables. The small parameters are identified as  $\epsilon$ , the maximum strain in the plate, and  $\frac{h}{\ell}$ , where  $h$  is the plate thickness and  $\ell$  is the characteristic length over which the deformation varies in the deformed plate. The variational-asymptotical method is then used, along with a Ritz-type approximation of three-dimensional displacement variables in the through-the-thickness coordinate, to approximate the three-dimensional strain energy of the plate with a function of two-dimensional quantities only. The above-mentioned Sturm-Liouville problem is identified, the eigenfunctions of which guarantee the warping to be of higher order than any retained degree of freedom if they are chosen as the shape functions for the retained degrees of freedom. The two-dimensional strain energy function is then given as a function of material constants and eigenfunctions.

We are able to calculate the exact solution for the Sturm-Liouville problem only for one- and two-layer plates. Thus, it was necessary to develop an approximate method of solution so that the plate theory could be finalized. We present a finite element analysis in one dimension (through the thickness) which we use to solve the Sturm-Liouville problem based on the shape functions of Ref. 14. Eigenvalues and eigenfunctions for various laminated plates obtained by a one-dimensional finite element method are presented and, when possible, compared with the exact solution. Application of the theory will be addressed in a later paper.

## Theoretical Development

The objective is to derive a strain energy function of a plate in terms of two-dimensional quantities only. It can be done only if some small parameters are present. We suppose that the parameters mentioned above,  $\epsilon$  and  $\frac{h}{\ell}$ , are small.

### Three-Dimensional Formulation

To begin we will formulate a three-dimensional development which shall be considered the exact solution to the plate problem.

**Undeformed State of Plate** A typical point in the undeformed plate can be located by introducing a Cartesian coordinate system  $x_i$  in such a way that  $x_\alpha: \{x_1, x_2\} \equiv x$  denotes

lengths along orthogonal straight lines in the mid-surface of the undeformed plate, and  $x_3 \equiv y \equiv h\zeta$  is the distance in the normal direction, where  $-\frac{1}{2} \leq \zeta \leq \frac{1}{2}$ . Throughout the analysis, Greek indices assume values 1 or 2; Latin indices assume values 1, 2, and 3; and repeated indices are summed over their ranges.

The spatial position vector  $\hat{r}(x_1, x_2, y)$  to an arbitrary point in the undeformed plate can be written as

$$\hat{r}(x, y) = r(x) + yb(x) \quad (1)$$

where  $r(x)$  is the spatial position vector of points on the mid-surface of a plate and  $b(x)$  is the unit normal vector. We will also need notation for unit vectors  $b_\alpha \equiv \frac{\partial r}{\partial x_\alpha}$  which, together with  $b$ , form an orthonormal triad.

Since the variable  $y$  (and  $\zeta$ ) is chosen specifically so that the spatial position vector  $r$  to a point on the reference mid-surface is the average position of points along the normal line at a particular value of  $x_1$  and  $x_2$ , then

$$r(x) = \langle \hat{r}(x, \zeta) \rangle \quad (2)$$

where the notation

$$\langle \bullet \rangle \equiv \frac{1}{h} \int_{-\frac{1}{2}}^{\frac{1}{2}} \bullet dy \equiv \int_{-\frac{1}{2}}^{\frac{1}{2}} \bullet d\zeta \quad (3)$$

is used throughout the paper.

**Deformed State of Plate** Without any restrictions the position vector  $\hat{R}(x_1, x_2, y)$  to an arbitrary point in the deformed plate can be represented by

$$\hat{R}(x, y) = R(x) + yB(x) + v_n(x, \zeta)B_n(x) \quad (4)$$

where  $R(x)$  is the position vector of points on the deformed reference surface,  $B_n(x)$ :  $\{B_1(x), B_2(x), B_3(x) \equiv B(x)\}$  is the reference orthonormal triad with vector  $B$  being orthogonal to the deformed reference surface, and  $v_n(x, \zeta)$  are components of the general warping displacement of an arbitrary point in the deformed normal line, consisting of both in- and out-of-plane components so that all possible deformations are considered.

The warping  $v_n$  could not be defined uniquely as a function of  $\zeta$  with an arbitrary choice of  $R(x)$  unless they are subject to the constraints

$$\langle v_n(x, \zeta) \rangle = 0 \quad (5)$$

which means that

$$R(x) = \langle \hat{R}(x, \zeta) \rangle \quad (6)$$

The orthogonality of vector  $B$  to the reference surface means

$$R_{,\alpha} \cdot B = 0 \quad (7)$$

where the notation  $(\cdot)_{,\alpha}$  denotes the partial derivative with respect to  $x_\alpha$ .

In order to eliminate the arbitrary rotation of vectors  $B_\alpha$  around normal  $B$  we impose the following constraint

$$B_1 \cdot R_{,2} = B_2 \cdot R_{,1} \quad (8)$$

A schematic of the plate deformation is shown in Fig. 1.

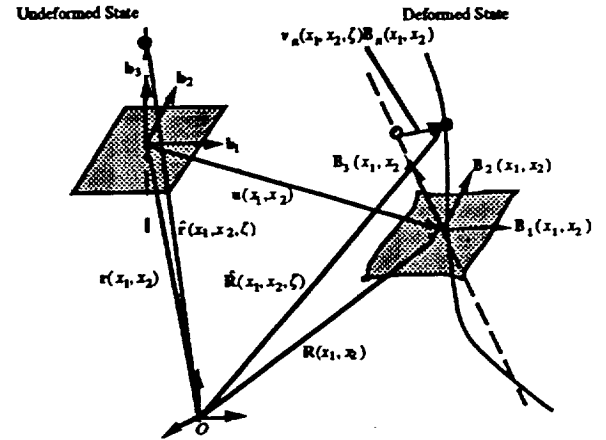


Fig. 1: Schematic of plate deformation ( $u$  is the displacement of the reference surface)

Thus, Eq. (4) provides a convenient way to represent the arbitrary function  $\hat{R}(x, y)$ . The orientation of the triad  $B_n$  is now specified uniquely.

**Strain Field** As shown in Ref. 6, under the condition of small local rotation, the Jaumann strain components can be arranged in a  $3 \times 3$  symmetric matrix  $\Gamma^*$ , given by

$$\Gamma^* = \frac{1}{2}(\chi + \chi^T) - I \quad (9)$$

$$\chi_{ij} = B_i \cdot \frac{\partial \hat{R}}{\partial x_j}$$

where  $I$  is the  $3 \times 3$  identity matrix.

Substituting Eq. (4) into the Jaumann strain, Eq. (9), one can express the strain field as a  $6 \times 1$  column matrix  $\Gamma = [\Gamma_{11}^* \ 2\Gamma_{12}^* \ \Gamma_{22}^* \ 2\Gamma_{13}^* \ 2\Gamma_{23}^* \ \Gamma_{33}^*]^T$  so that

$$\Gamma = \frac{1}{h} \Gamma_h v' + \Gamma_\epsilon \gamma + \Gamma_\ell v_x \quad (10)$$

where  $\epsilon \equiv \frac{\partial \gamma}{\partial x}$  and matrices  $\Gamma_A$ ,  $\Gamma_\epsilon$  and  $\Gamma_I$  are

$$\Gamma_A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \Gamma_\epsilon = \begin{bmatrix} I & \zeta I \\ 0 & 0 \end{bmatrix} \quad \Gamma_I = [\Gamma_1 \quad \Gamma_2] \quad (11)$$

$$\Gamma_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \Gamma_2 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\gamma = \begin{Bmatrix} \epsilon \\ hK \end{Bmatrix} \quad v_x = \begin{Bmatrix} v_{,1} \\ v_{,2} \end{Bmatrix} \quad (12)$$

Here we will introduce the two-dimensional strain measures for inplane stretching and shear

$$\epsilon = \begin{Bmatrix} R_{,1} \cdot B_1 - 1 \\ R_{,1} \cdot B_2 + R_{,2} \cdot B_1 \\ R_{,2} \cdot B_2 - 1 \end{Bmatrix} \quad (13)$$

and the two-dimensional strain measures for bending and twist

$$K = \begin{Bmatrix} B_{,1} \cdot B_1 \\ B_{,1} \cdot B_2 + B_{,2} \cdot B_1 \\ B_{,2} \cdot B_2 \end{Bmatrix} \quad (14)$$

The small parameter  $\epsilon$  can be now specified as

$$\epsilon = \max \|\gamma\| \quad (15)$$

A few nonlinear terms in the strain field, which couple  $v$  and  $\gamma$ , have been neglected in Eq. (10) because the physically linear plate theory is considered. The form of the strain field is of great importance because it is now linear in  $\gamma$ ,  $v'$ , and  $v_{,x}$ .

This is the only point where  $\epsilon$  as a small parameter is taken into account.

**Strain Energy of Plate** The strain energy density per unit area for a plate can be written as

$$U = \frac{1}{2} \langle \Gamma^T D \Gamma \rangle \quad (16)$$

where  $D$  is the  $6 \times 6$  symmetric material matrix in the  $B_n$  basis.

The three-dimensional Jaumann stress  $Z$ , which is conjugate to the Jaumann strain  $\Gamma$  is

$$Z = D\Gamma \quad (17)$$

## Basic Three-Dimensional Problem

The basic three-dimensional problem can now be represented as the following minimization problem

$$\int U(\gamma(x), v'(x, \zeta), v_{,x}(x, \zeta)) h dx_1 dx_2 + \text{terms with external forces} \rightarrow \min \quad (18)$$

where the minimum should be found with respect to three arbitrary functions  $R(x)$ , through which  $\gamma(x)$  is calculated, and to three functions  $v_n(x, \zeta)$  which are subject to the constraints of Eq. (5).

Note that the orthonormal triad  $B_n$  is not an independent variable, since it is a subject to the constraints in Eqs. (7) and (8). It is completely determined by the function  $R(x)$ .

## Dimensional Reduction

**Splitting of the Problem** Now the functional space of all the admitted functions  $\hat{R}(x, y)$  is split into sublayers with a choice of the  $x$ -dependent functions  $R(x)$ . In each layer the functions  $v_n(x, \zeta)$  are arbitrary under the constraint in Eq. (5).

We can solve this problem in two steps. First we are going to find functions  $v_n(x, \zeta)$  for any prescribed choice of functions  $R(x)$ . As a result, we will have  $v_n(x, \zeta)$  as a functional of  $R(x)$  and  $\zeta$ , and the functional, Eq. (18), will become dependent only on  $R(x)$ . That functional will give us a two-dimensional plate theory. The second step is to solve that theory.

Since the energy density  $U$  depends not only on functions  $v_n(x, \zeta)$  but also on their derivatives with respect to  $x$ , then the result of the first step will be very complicated (it will contain a non-local dependence on  $R(x)$  in the general case) and cannot be obtained in an appropriate form unless we take advantage of some small parameters.

**Small Parameters** Let us consider the situation where parameters  $h$ ,  $\ell$  and  $\epsilon$  are present. Since no coefficient matrix of  $\Gamma$ , Eq. (11), depends on these parameters, it is clear that the first term of the expression for strain, Eq. (10), has order  $\frac{\|v\|}{h}$ , the second term has order  $\epsilon$ , and the third one has order  $\frac{\|v\|}{\ell}$ . The third term has order  $\frac{h}{\ell}$  times that of the first. We should neglect this as a higher order term in the first approximation if we are going to expand the solution with respect to the small parameter  $\frac{h}{\ell}$ . In the future, this important circumstance allows us to avoid the presence of derivatives of unknown functions  $v_n(x, \zeta)$  with respect to  $x$  in the problem for any approximation and, hence, to solve it in an appropriate form.

Since our main problem has become linear with respect to the unknown functions  $v_n(x, \zeta)$  and the two-dimensional strain measure  $\gamma$ , the smallness of parameter  $\epsilon$  does not



need to be considered any more. This fact has already taken into account (see above under *Strain Field*). We will expand the warping  $v_\alpha(x, \zeta)$  as a series with respect to the small parameter  $\frac{1}{h}$  only. In the absence of other small parameters, expansion in  $\frac{1}{h}$  is the same as expansion in  $h$ . That is why we can consider  $h$  to be the small parameter in spite of its dimension.

### Discretization

The problem may be solved numerically by discretizing with respect to the thickness variable  $\zeta$ . Now the unknown functions  $v_\alpha(x, \zeta)$  are represented as the product of a matrix of shape functions  $S(\zeta)$  and a column matrix of nodal values of  $v(x, \zeta)$ , which we will denote  $V$

$$v(x, \zeta) = S(\zeta)V(x) \quad (19)$$

Substituting the discretized unknown function in Eq. (19) into the energy density, Eq. (16), while taking into account the strain, Eq. (10), one obtains

$$2U = \left(\frac{1}{h}\right)^2 V^T EV + \left(\frac{1}{h}\right) 2 V^T [D_{he}\gamma + D_{hz}V_x] + (1) (\gamma^T D_{ee}\gamma + 2 V_x^T D_{ze}\gamma + V_x^T D_{zz}V_x) \quad (20)$$

in which the following definitions are introduced

$$\begin{aligned} E &\triangleq \langle [\Gamma_h S]^T D [\Gamma_h S] \rangle \\ D_{he} &\triangleq \langle [\Gamma_h S]^T D [\Gamma_e] \rangle \\ D_{hz} &\triangleq \langle [\Gamma_h S]^T D [\Gamma_z S] \rangle \\ D_{ee} &\triangleq \langle [\Gamma_e]^T D [\Gamma_e] \rangle \\ D_{ze} &\triangleq \langle [\Gamma_z S]^T D [\Gamma_e] \rangle \\ D_{zz} &\triangleq \langle [\Gamma_z S]^T D [\Gamma_z S] \rangle \end{aligned} \quad (21)$$

### Classical Considerations

According to the variational-asymptotical procedure, in order to get the next approximation, one should retain only the leading energy term with respect to the small parameter that contains the unknown functions and the leading intersection term between the unknown function and the rest of the functional (for more details see Ref. 7).

We are left with the following expression

$$\left(\frac{1}{h}\right)^2 V^T EV + \left(\frac{1}{h}\right) 2 V^T D_{he}\gamma \quad (22)$$

This function should be minimized with respect to variable  $V$  under the constraint, Eq. (5), which is transformed to the following condition after discretization

$$V^T H \Psi_\alpha = 0 \quad H \triangleq \langle S^T S \rangle \quad (23)$$

where  $\Psi_\alpha$  is matrix with three columns, each corresponding to one of the constraints of Eq. (5). The set of columns  $\Psi_\alpha$  are determined by the kernel (null-space) of matrix  $E$  (for more details see Ref. 7). This means

$$E \Psi_\alpha = 0 \quad (24)$$

Let us suppose that the set of columns  $\Psi_\alpha$  is normalized in such a way that

$$\Psi_\alpha^T H \Psi_\alpha = I \quad (25)$$

The Euler equation for the problem posed by Eqs. (22) and (23) is

$$\left(\frac{1}{h}\right) EV + D_{he}\gamma = H \Psi_\alpha \mu \quad (26)$$

where  $\mu$  is the column matrix of Lagrange multipliers for the constraint in Eq. (23). By pre-multiplying Eq. (26) by  $\Psi_\alpha^T$  one can prove that

$$\mu = \Psi_\alpha^T D_{he}\gamma \quad (27)$$

Now the equation for  $V$ , Eq. (26), is rewritten as

$$\left(\frac{1}{h}\right) EV = -(I - H \Psi_\alpha \Psi_\alpha^T) D_{he}\gamma \quad (28)$$

Since  $E$  has a kernel,  $E^{-1}$  does not exist. However, the pseudo-inverse of  $E$ ,  $E_{cl}^+$ , satisfied the following relations

$$\begin{aligned} EE_{cl}^+ &= I - H \Psi_\alpha \Psi_\alpha^T \\ E_{cl}^+ E &= I - \Psi_\alpha \Psi_\alpha^T H \\ E_{cl}^+ EE_{cl}^+ &= E_{cl}^+ \end{aligned} \quad (29)$$

can be found (see the Appendix) and the solution of Eq. (28) can be represented as

$$V = -h E_{cl}^+ D_{he}\gamma \quad (30)$$

Substituting the solution, Eq. (30), into the discretized strain energy density, Eq. (20), and keeping only terms with the lowest order, which are equal to  $h^0 \equiv 1$  one obtains

$$2U = \gamma^T A \gamma \quad (31)$$

with

$$A \triangleq D_{ee} - [D_{he}]^T E_{cl}^+ [D_{he}] \quad (32)$$

The third property, Eq. (29), is taken into account here.

This is identical to the classical result for the strain energy of laminated plates.

### New Degrees of Freedom

In order to make our plate functional more flexible with respect to the variable  $x$ , let us introduce the new unknown plate functions such that

$$V(x) = \Psi_q q(x) + W(x) \quad (33)$$

where  $q$  is a column matrix of several new unknown functions,  $\Psi_q$  is a matrix, each column of which represents a  $\zeta$ -shape form associated with one of the new unknown functions  $q(x)$ , which will be named "new degrees of freedom," and  $W$  is the new warping to be found.

We will suppose that matrix  $\Psi_q$  is normalized in such a way that

$$\Psi_q^T H \Psi_q = I \quad (34)$$

The following constraint for  $W$  will make the splitting, Eq. (33), unique

$$W^T H \Psi_q = 0 \quad (35)$$

The order of functions  $q(x)$  with respect to  $h$  may be arbitrary and it will be supposed to be equal to  $h^0 \equiv 1$ .

As a  $\zeta$ -shape form of new degrees of freedom let us take eigenvectors of matrix  $E$  which correspond to the several lowest non-zero eigenvalues. Such a matrix  $\Psi_q$  will satisfy the following equation

$$E \Psi_q = H \Psi_q \Lambda_q \quad (36)$$

where  $\Lambda_q$  is a diagonal matrix of eigenvalues

$$\Lambda_q = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_{N_q} \end{bmatrix} \quad (37)$$

The constraint of Eq. (23), which still might be satisfied by  $W$ , can be combined with the constraint of Eq. (35) after introducing the matrix  $\Psi_u = [\Psi_{cl} \Psi_q]$  in such a way that

$$W^T H \Psi_u = 0 \quad (38)$$

Analogously, Eq. (36) can be rewritten as

$$E \Psi_u = H \Psi_u \Lambda_u \quad (39)$$

where matrix  $\Lambda_u$  is

$$\Lambda_u = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \lambda_1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \lambda_{N_q} \end{bmatrix} \quad (40)$$

### Calculation of Strain Energy

Let us assume that we have the correct expansion of  $V$  through order  $h^2$

$$V = V_0 + h V_1 + h^2 V_2 \quad (41)$$

where  $V_0$  denotes the first term of Eq. (33)

$$V_0 = \Psi_q q \quad (42)$$

The vector  $V_0$  satisfies the equation

$$E V_0 = H \Psi_q \Lambda_q q \quad (43)$$

and vectors  $V_1$  and  $V_2$  have to satisfy the constraint found in Eq. (38).

If we have an asymptotically correct expansion for Eq. (41), we can calculate an asymptotically correct energy for order  $h^0 \equiv 1$

$$\begin{aligned} 2U = & \left(\frac{1}{h}\right)^2 V_0^T E V_0 + \\ & + \left(\frac{1}{h}\right) 2 V_0^T [ \underline{E V_1} + D_{he} \gamma + D_{hz} V_{0,z} ] + \\ & + (1) \{ V_1^T E V_1 + 2 \underline{V_2^T E V_0} + \\ & + 2 V_1^T [ D_{he} \gamma + D_{hz} V_{0,z} ] + 2 V_{1,x}^T D_{zx} V_0 + \\ & + \gamma^T D_{ee} \gamma + 2 V_{0,x}^T D_{ze} \gamma + V_{0,x}^T D_{zz} V_{0,x} \} \end{aligned} \quad (44)$$

The underlined terms are equal to zero here because of Eq. (43) for  $V_0$  and the constraint of Eq. (38) for  $V_1$  and  $V_2$ . This means we do not need to know the second approximation for  $V$  (i.e.,  $V_2$ ) in order to calculate the energy for order  $h^0$ .

We shall minimize the functional

$$V_1^T E V_1 + 2 V_1^T [ D_{he} \gamma + (D_{hz} - D_{zh}^*) V_{0,z} ] \quad (45)$$

in order to find  $V_1$ .

The notation  $D_{zh}^*$  means

$$D_{zh}^* = [ (D_{hz1})^T \quad (D_{hz2})^T ] \quad (46)$$

which comes from  $2 V_{1,x}^T D_{zx} V_0$  in the fourth line of Eq. (44) after integration by parts with respect to  $x_1$  and  $x_2$ .

The Euler equation for the functional of Eq. (45) is

$$EV_1 + D_{h\epsilon}\gamma + (D_{hz} - D_{zh}^*)V_{0,z} = H\Psi_u\mu_u \quad (47)$$

where  $\mu_u$  is the Lagrange multiplier which enforces the constraint in Eq. (38).

Applying a procedure similar to the one used for the classical case we can calculate the Lagrange multiplier  $\mu_u$

$$\mu_u = \Psi_u^T [D_{h\epsilon}\gamma + (D_{hz} - D_{zh}^*)V_{0,z}] \quad (48)$$

and represent the solution of Eq. (47) by

$$V_1 = E_u^+ [D_{h\epsilon}\gamma + (D_{hz} - D_{zh}^*)V_{0,z}] \quad (49)$$

where the matrix  $E_u^+$  can be found with the following properties

$$\begin{aligned} EE_u^+ &= I - H\Psi_u\Psi_u^T \\ E_u^+E &= I - \Psi_u\Psi_u^TH \\ E_u^+EE_u^+ &= E_u^+ \end{aligned} \quad (50)$$

See the Appendix for an explanation of how to calculate the matrix  $E_u^+$ .

Substituting this expression into the energy, Eq. (44), one obtains

$$\begin{aligned} 2U = & \left(\frac{1}{h}\right)^2 V_0^T EV_0 + \left(\frac{1}{h}\right) 2 V_0^T [D_{h\epsilon}\gamma + D_{hz}V_{0,z}] + \\ & + (1)\{\gamma^T A_{\epsilon\epsilon}\gamma + 2 V_{0,z}^T P_{z\epsilon}\gamma + V_{0,z}^T P_{zz}V_{0,z}\} \end{aligned} \quad (51)$$

where the following notations are introduced

$$\begin{aligned} A_{\epsilon\epsilon} &\triangleq D_{\epsilon\epsilon} - [D_{h\epsilon}]^T E_u^+ [D_{h\epsilon}] \\ P_{z\epsilon} &\triangleq D_{z\epsilon} - [D_{hz} - D_{zh}^*]^T E_u^+ [D_{h\epsilon}] \\ P_{zz} &\triangleq D_{zz} - [D_{hz} - D_{zh}^*]^T E_u^+ [D_{hz} - D_{zh}^*] \end{aligned} \quad (52)$$

Finally, after substituting the expression for  $V_0$ , Eq. (42), the strain energy can be written as

$$2U = \begin{Bmatrix} \gamma \\ \frac{1}{h}q \\ q,z \end{Bmatrix}^T \begin{bmatrix} A_{\epsilon\epsilon} & A_{\epsilon q} & A_{\epsilon z} \\ A_{q\epsilon} & A_{qq} & A_{qz} \\ A_{z\epsilon} & A_{zq} & A_{zz} \end{bmatrix} \begin{Bmatrix} \gamma \\ \frac{1}{h}q \\ q,z \end{Bmatrix} \quad (53)$$

where

$$\begin{aligned} A_{qq} &\triangleq \Psi_q^T E \Psi_q = \Lambda_q \\ A_{qz} &\triangleq [\Psi_q^T D_{hz_1} \Psi_q \quad \Psi_q^T D_{hz_2} \Psi_q] = A_{qz}^T \\ A_{q\epsilon} &\triangleq \Psi_q^T D_{h\epsilon} = A_{\epsilon q}^T \\ A_{zz} &\triangleq \begin{bmatrix} \Psi_q^T P_{z_1 z_1} \Psi_q & \Psi_q^T P_{z_1 z_2} \Psi_q \\ \Psi_q^T P_{z_2 z_1} \Psi_q & \Psi_q^T P_{z_2 z_2} \Psi_q \end{bmatrix} \\ A_{\epsilon z} &\triangleq [P_{\epsilon z_1} \Psi_q \quad P_{\epsilon z_2} \Psi_q] = A_{z\epsilon}^T \end{aligned} \quad (54)$$

Eq. (53) represents the strain energy of a laminated plate undergoing deformation which is constrained only in the sense that the strain is small. Displacement and rotation of the normal line element appear nonlinearly in the expressions for  $\gamma$ . On the other hand, the new degrees of freedom give rise to simple linear Euler equations. Since the displacement field is now completely specified in terms of  $\gamma$ ,  $q$ , and  $q,z$ , it becomes a simple matter to recover strain and stress throughout the plate by use of Eqs. (10) and (17). The classical plate energy can be obtained from Eq. (53) after it has been minimized with respect to variables  $q$  with partial derivatives  $q,z$  being equal to zero.

## Numerical Results

Exact solutions for the Sturm-Liouville problem can be obtained with symbolic manipulation software, but we were only able to carry this out for one- and two-layer plates. Thus, we turned to a finite element solution. In Ref. 15 results for Sturm-Liouville problems with discontinuous coefficients were obtained which agree quite well with the exact solution. Our finite element method is similar except that the orthogonal Jacobi-polynomial-based shape functions presented in Ref. 14 are used. This means that our finite elements have interior degrees of freedom which can be added without generation a new element geometry. This mesh through the thickness can be as fine as we wish, but we must at least have element breaks where discontinuities exist. The highest derivatives involved in the problem are first derivatives with respect to  $\zeta$ , and thus  $C^0$  continuous shape functions can be used. See Ref. 14 for details.

In this section we present some numerical results for the solutions of the Sturm-Liouville problem, compared when possible with the exact solution. We start with an error analysis of the eigenvalues for the two-layer case, and we conclude with the eigenfunctions of one-, two-, and four-layer plates. The four-layer results include both a symmetric plate and a non-symmetric cross-ply plate.

For the purpose of discussion only, we align our coordinate axes with  $x_1$  along the length ("longitudinal inplane") and  $x_2$  along the width ("lateral inplane"); these are not necessarily aligned with any material axes. For our examples, we choose a fiber reinforced composite material which has the following material properties

$$\begin{aligned} E_L &= 25 \times 10^6 \text{ psi} & E_T &= 10^6 \text{ psi} \\ G_{LT} &= 0.5 \times 10^6 \text{ psi} & G_{TT} &= 0.2 \times 10^6 \text{ psi} \\ \nu_{LT} &= \nu_{TT} &= 0.25 \end{aligned}$$

where signifies the direction parallel to the fibers and  $T$  the transverse direction. These properties, along with a ply angle, allow the calculation of the matrix  $D$ . Note, however, that in the example problems and indeed in all laminated plates, certain terms in  $D$  will vanish. Our theory does not require any special terms in  $D$  to be zero. Because of the vanishing terms in the example problems, there is no coupling between out-of-plane and inplane displacement components. Thus, certain modes will be entirely inplane

and others entirely out-of-plane. Also, since  $D_{66}$  is constant through the thickness for such plates, the thickness mode with the lowest eigenvalue for all the examples will be a sine function.

As examples we consider the following four lay-ups:

[15°]	"one-layer"
[-15°/15°]	"two-layer"
[30°/-30°] <sub>sym</sub>	"four-layer"
[0°/90°/0°/90°]	"cross-ply"

where the words in quotation marks indicate the terms we use to designate the plate under consideration.

### Eigenvalues for Two-Layer Plate

All of the plates have three zero eigenvalues. These correspond with the classical "degrees of freedom" of the normal line element, the average displacement components of that line element. The eigenvalues appear in the above equations as well, and the size of these eigenvalues may have a bearing on whether the associated degree of freedom is an important one. Our present understanding is that the smallest eigenvalues are the ones of interest, but this must be confirmed by application of the theory with different choices for the degrees of freedom. Here we will present a few of the smallest non-zero eigenvalues for the two-layer plate and the analysis of the error, since the exact solution is available for this case.

In Table 1 the first four nonzero eigenvalues are shown for the two-layer plate, both from the exact solution and from our finite element approximation with four elements ( $M=4$ ). The order of the shape functions is varied by changing  $J$ , the number of Jacobi polynomials used to construct the higher-order shape functions. The crudest element has  $J=0$ , resulting in linear shape functions. Results are shown for  $J=1$  (quadratic shape functions) and  $J=2$  (cubic shape functions). It is seen that all of the finite element results are very close to the exact solution. Furthermore, the higher-order shape functions greatly improve the accuracy.

Exact	Numerical Results ( $M=4$ )		
	$J=0$	$J=1$	$J=2$
2.04599617	2.15317042	2.04710246	2.04600108
4.29843523	4.54270883	4.30275294	4.29845836
8.87418665	10.84944490	8.93773416	8.87567158
10.57173730	11.12555748	10.57151555	10.57176142

Table 1: Eigenvalues for 2-layer plate

A more precise analysis of the error for the lowest nonzero eigenvalue is presented in Figs. 2 and 3. In Fig. 2 the relative error versus the size of the eigenvalue problem is essentially a straight line on a log-log scale if  $J$ , the number of Jacobi polynomials in each element, is held constant. This means that the size of the matrix increases as the number of elements through the thickness is increased. In Fig. 3

the relative error versus the size of the eigenvalue problem is shown for the case when the mesh is held constant; here the size of the matrix increases as the order of element shape functions is increased. We will present our results below for the eigenfunctions based on elements with cubic shape functions. To analyze plates with a large number of layers, the number of elements must be increased. If only the higher-order functions were used, such analyses could become expensive. However, as seen in these plots, an increase in the number of elements, which would be required because of discontinuities in the properties between layers, would bring the error down without the need of higher-order shape functions.

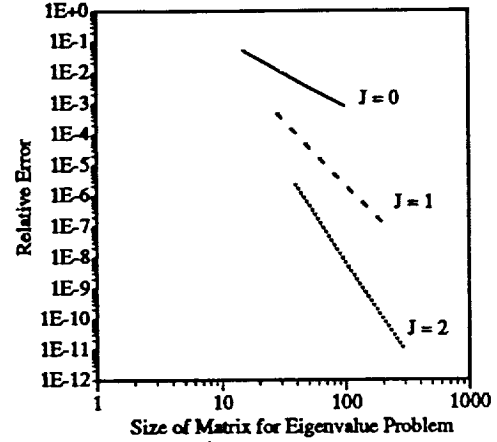


Fig. 2: Relative error for  $\lambda_1$  of 2-layer plate versus size of matrix  $E$  for constant  $J$

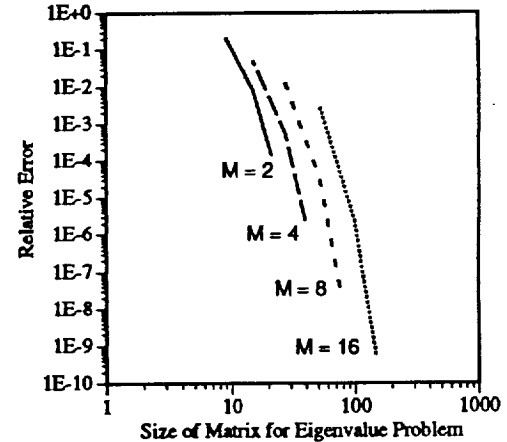


Fig. 3: Relative error for  $\lambda_1$  of 2-layer plate versus size of matrix  $E$  for constant  $M$

### Eigenfunctions for One-Layer Plate

Eigenfunctions for the smallest four nonzero eigenvalues of the one-layer plate are presented in Figs. 4a - 4d. The finite element results, shown as solid lines, were obtained using 2 elements with cubic shape functions; and the exact solution, shown as dashed lines, is indistinguishable

from the finite element results in the plots. Fig. 4a shows the eigenfunction for a transverse shear mode dominated by the lateral inplane displacement, while Fig. 4b shows the corresponding one dominated by the longitudinal inplane displacement. Note in Fig. 4a the two displacement components are of opposite sign while in Fig. 4b they are of the same sign. Fig. 4c shows a higher mode of the same type as shown in Fig. 4a. The fourth nonzero eigenvalue has an eigenfunction with only out-of-plane displacement. We note the smooth character of these functions, which are close to sines and cosines. Plate theories based on expansion of the displacement in power series or trigonometric should provide excellent predictive capability for homogeneous plates such as this one.

### Eigenfunctions for Two-Layer Plate

Eigenfunctions for the smallest four nonzero eigenvalues of the two-layer plate are presented in Figs. 5a – 5d, along with the corresponding eigenvalues. The finite element results were obtained using 4 elements with cubic shape functions, and the exact solution is again indistinguishable from the finite element results in the plots. Fig. 5a shows the eigenfunction for a transverse shear mode dominated by the lateral inplane displacement. Fig. 5b shows the corresponding mode having more longitudinal inplane displacement, but not so strongly dominated by it. Fig. 5c shows a higher mode of the same type as shown in Fig. 5a. The fourth nonzero eigenvalue again has an eigenfunction with only out-of-plane displacement, as shown in Fig. 5d.

Although the displacement functions shown in Figs. 5c and 5d both appear to be smooth, the displacement has a discontinuous slope in both Figs. 5a and 5b, in sharp contrast to results for the one-layer plate. Power series approximations through the entire thickness of the plate would not be able to capture this behavior, but the above theory shows that the warping induced by this type of displacement is a higher-order effect. Thus, a plate theory with these degrees of freedom should be a noticeable improvement over classical theory. Furthermore, such a theory should be an improvement over the theory of Refs. 9 and 10.

### Eigenfunctions for Four-Layer Plate

Eigenfunctions for the smallest four nonzero eigenvalues of the four-layer plate are presented in Figs. 6a – 6d. These results were obtained using 8 elements with cubic shape functions. Figs. 6a and 6b show the eigenfunctions for coupled transverse shear modes, the former having more lateral inplane displacement while the latter has more longitudinal inplane displacement. Fig. 6c shows a higher mode of the same type as shown in Fig. 6a. The fourth nonzero eigenvalue again has a smooth eigenfunction with only out-of-plane displacement, shown in Fig. 6d, as expected. Again, in sharp contrast to results for the one-layer plate, the displacement has discontinuous slopes in Figs. 6a – 6c. The symmetry of the plate is exhibited in the symmetry of the modes.

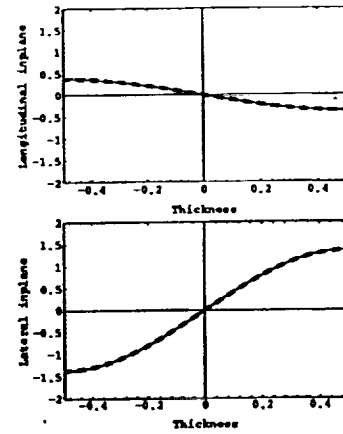


Fig. 4a:  $\lambda = 1.97392088$

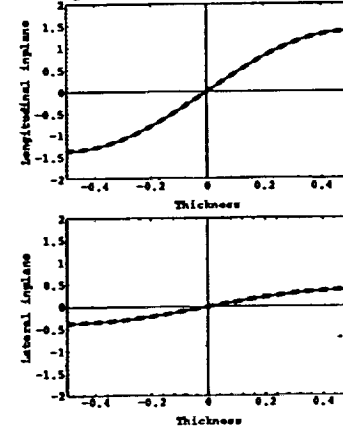


Fig. 4b:  $\lambda = 4.9348022$

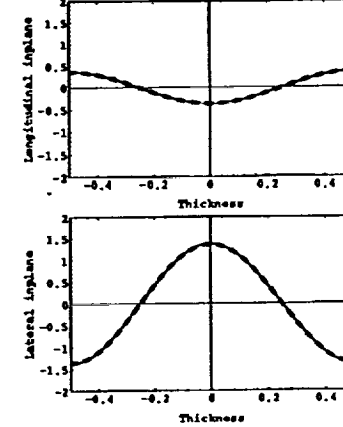


Fig. 4c:  $\lambda = 7.89568352$

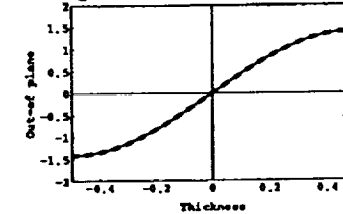


Fig. 4d:  $\lambda = 10.57173733$

Fig. 4: Eigenfunctions for 1-layer plate

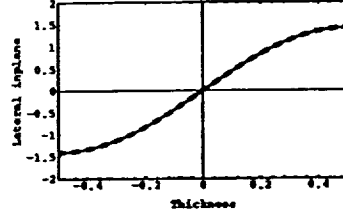
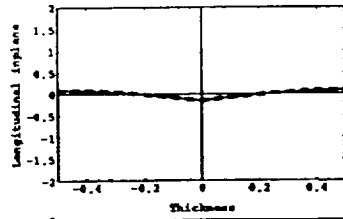


Fig. 5a:  $\lambda = 2.04599617$

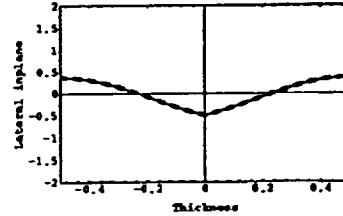
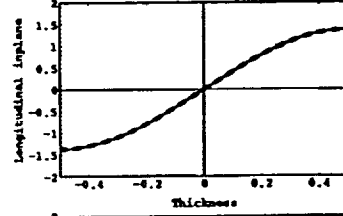


Fig. 5b:  $\lambda = 4.29843523$

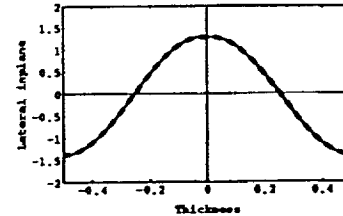
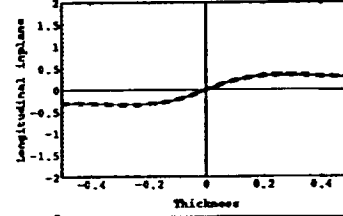


Fig. 5c:  $\lambda = 8.87418665$

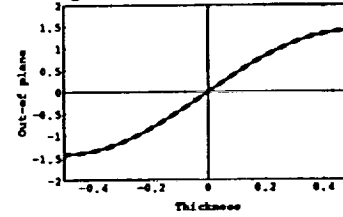


Fig. 5d:  $\lambda = 10.57173730$

Fig. 5: Eigenfunctions for 2-layer plate

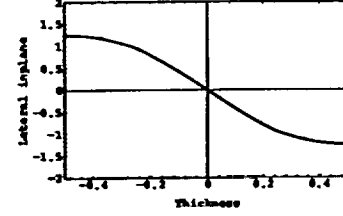
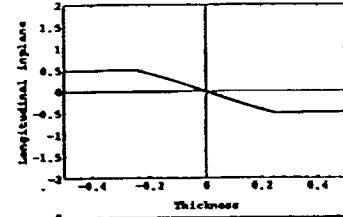


Fig. 6a:  $\lambda = 2.12930862$

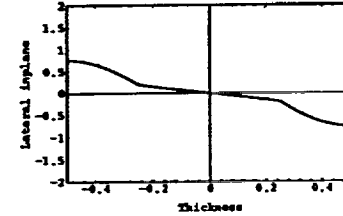
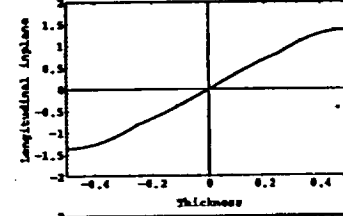


Fig. 6b:  $\lambda = 4.06979407$

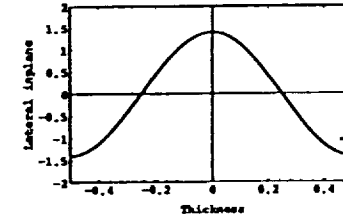
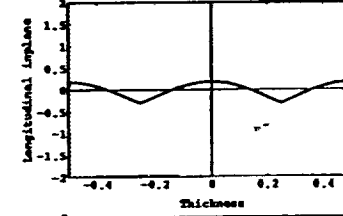


Fig. 6c:  $\lambda = 9.10254004$

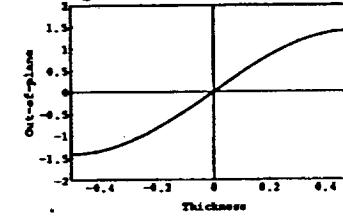


Fig. 6d:  $\lambda = 10.57173771$

Fig. 6: Eigenfunctions for 4-layer plate

## Eigenfunctions for Cross-Ply Plate

Eigenfunctions for the smallest four nonzero eigenvalues of the cross-ply plate are presented in Figs. 7a – 7d. These results were obtained using 8 elements with cubic shape functions. Figs. 7a and 7b show the eigenfunctions for coupled transverse shear modes (with the same eigenvalues), one having more lateral inplane displacement while the other has more longitudinal inplane displacement. The third, fourth, and fifth nonzero eigenvalues are also the same. The eigenfunctions corresponding to two of these eigenvalues are shown in Figs. 7c and 7d. The out-of-plane mode is shown in Fig. 7c, having the same smooth eigenfunction as before. Fig. 7d shows a higher mode of the same type as shown in Fig. 7b. The other eigenfunction for this triple root (not shown) is a higher mode of the type as shown in Fig. 7a. Again, in contrast to results for the one-layer plate, the displacement has discontinuous slopes – this time in Figs. 7a, 7b, and 7d. The lack of symmetry of the plate is exhibited in a lack of symmetry in the inplane modes.

It should be clear that choosing *a priori* displacement fields which exhibit the character of the inplane modes shown in Figs. 4 – 7 would be virtually impossible. They have the appropriate symmetry or asymmetry, as well as the discontinuities which reflect the layup.

## Concluding Remarks

A geometrically nonlinear theory for laminated plates is presented based on a combination of the variational-asymptotical method and the method of Ritz. The displacement field is described in terms of the average displacement of the normal line element and a small number of additional functions of the in-plane coordinates of the plate. The through-the-thickness shape functions for these new “degrees of freedom” are not analytic functions for arbitrarily laminated plates. Rather, they are eigenfunctions of a certain Sturm-Liouville problem based on the thickness coordinate of the plate. Unlike power series formulations, this allows for the correct treatment the known jumps in the stress and strain fields. Unlike layerwise variable theories, the present theory has only a small number of variables in addition to those found in classical plate theory, a number which does not depend on the number of layers in the plates. Additional equilibrium equations for the plate theory associated with the new degrees of freedom are simple, linear equations – even for a large-displacement theory.

Since analytical solutions of the Sturm-Liouville problem are limited to one- and two-layer plates, an approximate finite element solution was obtained. Results obtained for these shape functions are presented for a variety of laminated plate configurations. These results agree well with available exact solutions. In the future numerical studies will be conducted in order to determine how many and which types of degrees of freedom produce the best all-around plate theory.

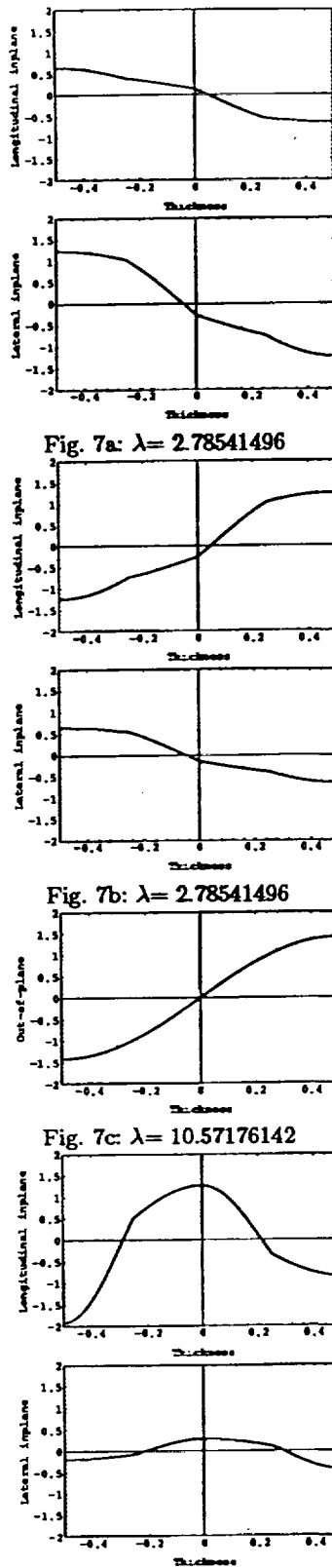


Fig. 7: Eigenfunctions for cross-ply plate

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## Appendix

### Calculation of Pseudo-Inverse Matrix

Let columns of matrix  $\Psi$  be the set of all eigenvectors. In other words the matrix  $\Psi$  satisfies the following system of equations

$$E\Psi = H\Psi\Lambda$$

$$\Psi^T H \Psi = I$$

where the diagonal matrix of eigenvalues  $\Lambda$  is

$$\Lambda = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \lambda_1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \lambda_{N_e} & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & \lambda_{N_e+1} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & \lambda_{N-3} \end{bmatrix}$$

where  $N$  is the dimension of matrix  $E$ .

Now, it is easy to check that Eq. (50) and the following expressions are true

$$E = H\Psi\Lambda\Psi^T H$$

$$\Psi\Psi^T = H^{-1}$$

$$E_{\tau}^{+} = \Psi\Lambda_{\tau}^{-1}\Psi^T$$

where  $\Lambda_{\tau}^{-1}$  is

$$\Lambda_{\tau}^{-1} = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & \frac{1}{\lambda_{N_e+1}} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & \frac{1}{\lambda_{N-3}} \end{bmatrix}$$